SOME UNSOLVED PROBLEMS IN STATISTICAL DESIGN AND LINEAR MODEL THEORY

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Abstract

As might be expected in actively researched fields such as statistical design and linear model theory, many unresolved problems come to light. Seven such problems in areas relating to orthogonality in latin squares and F-squares, difference sets, sequential design, optimal design, and linear model theory, are discussed. The status of the problem is given along with partial solutions and some suggestions for obtaining solutions.

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0. Introduction. As any active researcher in statistical design theory well knows, a multitude of unsolved problems exist. These problems are of many and diverse kinds, ranging from the design of a particular medical trial for a repeated measurements experiment to determining the cardinality of singular $(0,1,\cdots,a-1)$-matrices. Some statisticians (and also some editors of statistical journals) insist that only research of the former type is important, while others argue for the latter type of research. All such research is important since the solution for a theoretical combinatorial problem may provide the answer for a particular design problem for an experimenter; numerous examples are available to substantiate this statement. Much research in statistical design requires the development of new mathematical theory. Mathematical research on orthogonality of latin, and more recently F-squares, and on block designs are now well accepted areas of research for mathematicians. One needs only to peruse the table of contents of such periodicals as the Journal of Combinatorial Theory and Utilitas Mathematica, for example, to verify this. This is one area in which research in mathematics and statistics overlaps. The solution of combinatoric problems often provide answers for statistical design problems.

An F-square of order n with m symbols is denoted as $F(n;\ \lambda_1,\lambda_2,\cdots,\lambda_m)$. The λ_i are integers and refer to the frequency with which a symbol occurs in any row or any column. When the $\lambda_i = 1$ and $n = m$, a latin square results. Note that $\Sigma_1^m \lambda_i = n$ for any F-square. A set of t mutually orthogonal F-squares with a common number of symbols is denoted as $OF(n;\lambda_1,\cdots,\lambda_m;\ t)$ to correspond with the notation used to denote a set of t mutually orthogonal latin squares (i.e., $OL(n,t)$) of order n. The number of symbols in a set of orthogonal F-squares may vary, and for this we use the notation $\Sigma_{i=2}^{n}\ OF(n;\lambda_1,\cdots,\lambda_i;N_{\lambda i})$ for all λ_h, $h = 1,2,\cdots,i$, to indicate that there are $N_{\lambda i}$ F-squares with i symbols for each possible set of the λ_h.

In the following we discuss seven unsolved problems. The first problem is to find a simple and easy-to-use algorithm for constructing a set of $t \leq n-1$ mutually orthogonal latin squares. The prime numbers case turns out to be simple. The prime powers case is partly resolved. The second problem is similar to the first, but it considers $n = 2(4t+2)$. Likewise, problem 3 is similar to problem 1, but it relates to $n = p^{\alpha}q^{\beta}$ and $n = r,s$ (r,s odd). An example with $n = 15$ is discussed.

Problem 4 is in a different vein in that it considers writing down the rows of a latin square in such a manner as to achieve variance-balance or as near variance-balance as is possible. Designs of this nature would be useful if rows were added sequentially and if a stopping rule with some optimality properties were in operation. The fifth problem deals with finding complete sets of orthogonal F-squares for various numbers of symbols and values of n. The sixth problem is in the area of linear model theory, and its solution would be useful in constructing experiment designs (plans for arrangement of the treatments in an experiment). Several other unresolved questions and a conjecture are posed. The last problem relates to the optimal design problem, researched by Elfving [1952, 1956], Kiefer [1959, 1961], Federov [1972], Wynn [1972], and several others, when the regression function goes through a known point (X_0, Y_0) either in or outside the factor-space.

1. Problem 1. The tabulation of complete sets of latin squares or F-squares of order n requires a considerable amount of space if n covers even a moderate range of values. Therefore, a simple method of construction and a space-conserving procedure is desired. For n a prime number, such a procedure does exist. A method denoted as "projecting diagonals" (see Federer *et al.* [1970]) consists of writing a latin square in standard order with the symbols in consecutive order, and then writing the symbols in the right diagonal of the preceding square as the first column of square two, with the remaining symbols being written in

consecutive order in rows. Repeat the procedure on square two to obtain square three, $\cdot\cdot\cdot$, until all n-1 squares are obtained. The procedure is illustrated for n = 3 and 5 below:

$$\begin{array}{ccc} \underline{\text{square 1}} & & \underline{\text{square 2}} \\ \begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{matrix} & \rightarrow & \begin{matrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{matrix} \end{array} = \text{OL}(3,2) \text{ set}$$

$$\begin{array}{ccccccc} \underline{\text{square 1}} & & \underline{\text{square 2}} & & \underline{\text{square 3}} & & \underline{\text{square 4}} \\ \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{matrix} & \rightarrow & \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \\ 5 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 & 1 \\ 4 & 5 & 1 & 2 & 3 \end{matrix} & \rightarrow & \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \\ 2 & 3 & 4 & 5 & 1 \\ 5 & 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 1 & 2 \end{matrix} & \rightarrow & \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \\ 4 & 5 & 1 & 2 & 3 \\ 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 \end{matrix} \end{array} = \text{OL}(5,4) \text{ set}$$

The above rule is sufficient for generating a complete set of orthogonal latin squares for n a prime number. The proof rests on Theorem 3.1 of Hedayat and Federer [1969]. Hence, an easy and concise method for obtaining a complete set of orthogonal latin squares is available for n a prime number.

For n a prime power, the problem is partly solved (Hedayat and Federer [1970] and J. P. Mandeli [1978]). For $n = p^m$, first construct a $p \times p$ cyclic latin square $L(p)$. Secondly construct a latin square of order $n = p^m$ as the Kronecker product of m latin squares of order p, i.e., $\Pi_{i=1}^{m} \otimes L_i(p) = L_0^*(n)$. Thirdly, order the rows of the above square according to an automorphism of order $n = p^m$ to produce $L_0(n)$. Fourthly, perform n-2 consecutive cyclic row permutations to obtain n-2 additional latin squares which together with $L_0(n)$ form a mutually orthogonal set, OL(n,n-1). To illustrate, consider the two squares $L_0(8)$ and $L_0(9)$ given by Hedayat and Federer [1970]. Let

$$L_0^*(8) = \prod_{i=1}^{3} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{bmatrix} 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \\ 001 & 000 & 011 & 010 & 101 & 100 & 111 & 110 \\ 010 & 011 & 000 & 001 & 110 & 111 & 100 & 101 \\ 011 & 010 & 001 & 000 & 111 & 110 & 101 & 100 \\ 100 & 101 & 110 & 111 & 000 & 001 & 010 & 011 \\ 101 & 100 & 111 & 110 & 001 & 000 & 011 & 010 \\ 110 & 111 & 100 & 101 & 010 & 011 & 000 & 001 \\ 111 & 110 & 101 & 100 & 011 & 010 & 001 & 000 \end{bmatrix} .$$

If the rows and columns of $L_0^*(8)$ are reordered as follows (000) = 1, (010) = 2, (001) = 3, (101) = 4, (111) = 5, (110) = 6, (011) = 7, and (100) = 8, the $L_0(8)$ square of Hedayat and Federer [1970] is obtained. This square plus cyclic row permutations are illustrated below:

$L_0(8)$	$L_1(8)$	$L_2(8)$		$L_6(8)$
1 2 3 4 5 6 7 8	1 2 3 4 5 6 7 8	1 2 3 4 5 6 7 8	...	1 2 3 4 5 6 7 8
2 1 7 5 4 8 3 6	8 6 2 3 7 2 5 1	7 3 2 6 8 4 1 5		3 7 1 8 6 5 2 4
3 7 1 8 6 5 2 4	2 1 7 5 4 8 3 6	8 6 4 3 7 2 5 1		4 5 8 1 2 7 6 3
4 5 8 1 2 7 6 3	3 7 1 8 6 5 2 4	2 1 7 5 4 8 3 6		5 4 6 2 1 3 8 7
5 4 6 2 1 3 8 7	4 5 8 1 2 7 6 3	3 7 1 8 6 5 2 4		6 8 5 7 3 1 4 2
6 8 5 7 3 1 4 2	5 4 6 2 1 3 8 7	4 5 8 1 2 7 6 3		7 3 2 6 8 4 1 5
7 3 2 6 8 4 1 5	6 8 5 7 3 1 4 2	5 4 6 2 1 3 8 7		8 6 4 3 7 2 5 1
8 6 4 3 7 2 5 1	7 3 2 6 8 4 1 5	6 8 5 7 3 1 4 2	...	2 1 7 5 4 8 3 6

These seven squares form an OL(8,7) set.

Likewise, if one lets $L_0^*(9) = \prod_{i=1}^{2} \otimes \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}_i$ and lets the row and column order be (00) = 1, (01) = 2, (11) = 3, (12) = 4, (20) = 5, (02) = 6, (22) = 7, (21) = 8, and (10) = 9, the $L_0(9)$ square of Hedayat and Federer [1970] is obtained. Then using a cyclic row permutation of the last eight rows of $L_0(9)$, one may obtain an OL(9,8) set. Note that the column order is irrelevant but that the above row ordering is crucial in order to use the row reshuffling or cyclic row permutation rule. This leads to the following theorem (J. P. Mandeli [1978]):

Theorem. $L_0(n=p^m)$, p a prime number, may be written as $\Pi_{i=1}^{m} \otimes L(p)_i$, where $\otimes$ denotes Kronecker product and $L(p)_i$ is a cyclic latin square of order p, after ordering the rows according to an automorphism.

The problem here is to present a simple, mathematically elementary procedure for writing the automorphisms of order $n=p^m$. If this problem were resolved, then the amount of space required for writing a rule for complete sets of orthogonal latin squares for primes and prime powers would be minimal and easily usable by a large number of individuals. The method has been simplified to some extent by Raktoe [1967] and Hedayat and Federer [1970] for $n = p^m \leq 1000$, but it needs to be made simpler in order for students and experimenters to make use of the method.

2. Problem 2. The second problem relates to constructing $s \leq n-1$ sets OL(n,s) for n = 12, 20, ⋯, 2(4t+2) and to finding a simple concise form for writing these sets. As was noted in Federer et al. [1970], the OL(12,5) set of Johnson et al. [1961] may be obtained from the Kronecker product of two squares as:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 5 & 0 & 1 & 2 & 3 & 4 \\ 4 & 5 & 0 & 1 & 2 & 3 \\ 3 & 4 & 5 & 0 & 1 & 2 \\ 2 & 3 & 4 & 5 & 0 & 1 \\ 1 & 2 & 3 & 4 & 5 & 0 \end{bmatrix}$$

Furthermore, we may write the OL(12,5) set as

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
5 & 6 & 3 & 10 & 2 \\
4 & 10 & 6 & 5 & 7 \\
3 & 4 & 5 & 7 & 8 \\
2 & 11 & 9 & 1 & 4 \\
1 & 5 & 7 & 9 & 11 \\
6 & 9 & 4 & 3 & 10 \\
11 & 7 & 10 & 8 & 6 \\
10 & 2 & 1 & 4 & 9 \\
9 & 8 & 2 & 11 & 5 \\
8 & 1 & 11 & 6 & 3 \\
7 & 3 & 8 & 2 & 1
\end{bmatrix}
$$

and by noting that these are the first columns of the five mutually orthogonal squares and that the remainder of any row of a square occurs in consecutive order within the left and right halves of the square. For example, consider square five from the fifth column above:

0	1	2	3	4	5	6	7	8	9	10	11
2	3	4	5	0	1	8	9	10	11	6	7
7	8	9	10	11	6	1	2	3	4	5	0
8	9	10	11	6	7	2	3	4	5	0	1
4	5	0	1	2	3	10	11	6	7	8	9
11	6	7	8	9	10	5	0	1	2	3	4
10	11	6	7	8	9	4	5	0	1	2	3
6	7	8	9	10	11	0	1	2	3	4	5
9	10	11	6	7	8	3	4	5	0	1	2
5	0	1	2	3	4	11	6	7	8	9	10
3	4	5	0	1	2	9	10	11	6	7	8
1	2	3	4	5	0	7	8	9	10	11	6

left half   right half

The above OL(12,5) set involves a row shuffling, but what is the rule for $n = 2(4t+2)$? Will an appropriate row shuffling produce $4t+1 = s$ mutually orthogonal squares in general? Can one use $L_i(3) \otimes L_i(4)$ to produce $L_0(12)$ in order to find an $s > 5$? A computer search for these problems is not recommended because this becomes impossible for $t > 1$.

3. Problem 3. Another problem is to construct $t \leq n-1$ mutually orthogonal latin squares of order $n = p^{\alpha}q^{\beta}$ for $q^{\beta} = p^{\alpha}+2$ (the twin primes case) and for $n = rs$, where r,s are odd integers, and to find a simple method of writing these. For example, an OL(15,3) set is presented in Federer et al. [1970] and by Hedayat [1971]. The set may be written as the column vector

$$[(0,0,0)(1,7,5)(2,1,7)(3,6,2)(4,11,12)(5,13,8)(6,2,1)(7,5,13)(8,10,4)$$
$$(9,14,3)(10,4,11)(11,12,9)(12,9,14)(13,8,10)(14,3,6)]'$$

together with the rule that the entries in the first row are in consecutive order of integers and the entries in the column are cyclic permutations of the order of entries in the first column. The three squares were generated by the following cyclic permutations:

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 1 & 11 & 10 & 7 & 9 & 14 & 13 & 6 & 0 & 3 & 4 & 2 & 8 & 5 & 12 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 7 & 4 & 8 & 9 & 11 & 2 & 5 & 12 & 13 & 10 & 0 & 14 & 1 & 3 & 6 \end{bmatrix}$$

The rows were reordered to have the first square in numerical order in the above column vector.

An apparently related result is given by Federer, Joiner, and Raghavarao [1974] in which the subscripts on the matrices in the 2^{nd} to next to last columns represent the first columns of an OL(n,n-1) set of orthogonal latin squares. Can this result be utilized to extend the OL(15,3) set?

4. Problem 4. In using a latin square design for repeated measurements through time, it would be desirable to add the rows of a latin square in a manner which achieves as near variance-balance as possible. That is, rows are added in such a manner as to form the nearest possible Youden design. For example, suppose that v = 7 treatments (symbols 0,1,···,6) are to be used in a latin square of order seven. If the rows are added in an appropriate manner, Youden designs are formed at various stages. This is illustrated below:

	Column						
Row	1	2	3	4	5	6	7
1	3	4	5	6	0	1	2
2	5	6	0	1	2	3	4
3	6	0	1	2	3	4	5
4	0	1	2	3	4	5	6
5	1	2	3	4	5	6	0
6	2	3	4	5	6	0	1
7	4	5	6	0	1	2	3
8	3	4	5	6	0	1	2

Rows 1-3 form a SBBD(v=b=7,r=k=3,λ=1)

Rows 1-4 form a SBBD(7,4,2)

Rows 5-7 form a SBBD(7,3,1)

Rows 1-6 form a SBBD(7,6,5)

Rows 1-8 form a SBBD(7,8,9)

where SBBD(v,k,λ) denotes a symmetrical balanced block design for v=b treatments and block sizes k=r, i.e., the so-called v,k,λ-configurations.

If an appropriate sequential stopping rule were available, statistical analyses of the above design could be made when three, four, six, seven, eight, etc. rows have been completed and a variance-balanced design would have resulted. Whenever experiment designs of the above nature are constructed, it will be up to the sequential analysis researcher to provide appropriate stopping rules and analyses.

As another example consider the following latin square of order 13 with treatments 0,1,•••,12:

	Column												
Row	1	2	3	4	5	6	7	8	9	10	11	12	13
1	3	4	5	6	7	8	9	10	11	12	0	1	2
2	9	10	11	12	0	1	2	3	4	5	6	7	8
3	11	12	0	1	2	3	4	5	6	7	8	9	10
4	12	0	1	2	3	4	5	6	7	8	9	10	11
5	5	6	7	8	9	10	11	12	0	1	2	3	4
6	7	8	9	10	11	12	0	1	2	3	4	5	6
7	8	9	10	11	12	0	1	2	3	4	5	6	7
8	6	7	8	9	10	11	12	0	1	2	3	4	5
9	0	1	2	3	4	5	6	7	8	9	10	11	12
10	1	2	3	4	5	6	7	8	9	10	11	12	0
11	2	3	4	5	6	7	8	9	10	11	12	0	1
12	4	5	6	7	8	9	10	11	12	0	1	2	3
13	10	11	12	0	1	2	3	4	5	6	7	8	9

Rows 1 - 4 form a SBBD(13,4,1)

Rows 10 - 13 form a SBBD(13,4,1)

Rows 1 - 9 form a SBBD(13,9,6)

Rows 1 - 12 form a SBBD(13,12,11)

A third example is for the SBBD(v=31,k,λ) designs. Let the initial entries in the first ten rows be 7, 9, 13, 14, 20, 2, 6, 16, 21, 23, with the entries in each row being numbered consecutively. The resulting first six rows form a SBBD(31,6,1) and the resulting first ten rows form a SBBD(31,10,3). It should be possible to construct a latin square of order 31 such that

Rows 1 - 6 form a SBBD(31,6,1)

Rows 1 - 10 form a SBBD(31,10,3)

Rows 1 - 15 form a SBBD(31,15,7)

Rows 1 - 16 form a SBBD(31,16,8)

Rows 17 - 31 form a SBBD(31,15,7)

Rows 1 - 30 form a SBBD(31,30,29)

We know that for $n = 4t+3$ the quadratic residues form a difference set with $(n-1)/2$ elements. The $(n+1)/2$ nonquadratic residues likewise form a set. The zero element can be eliminated from the latter set leaving $(n-1)/2$ elements in the set. For example, 1, 2, and 4 are the quadratic residues for $n = 7$ and 0, 3, 5, and 6 are the nonquadratic residues. Eliminating the zero element, the values 3, 5, and 6 can be used to form a SBBD(7,3,1). Now, what can one do for any n? Also, what are the necessary and sufficient conditions for a subset of a difference set to be a difference set itself? Once these problems are resolved one can proceed to the sequential stopping rules and statistical analyses.

5. Problem 5. Complete sets of mutually orthogonal F-squares for $n = s^m$, s a prime power have been presented by Hedayat *et al.* [1975] and by Mandeli [1975]. In addition, complete sets of mutually orthogonal F-squares for $n = 4t$ with two equally replicated symbols and for all t for which a Hadamard matrix exists, have been shown to exist (Federer [1977]) and are easily constructed. Using some results from Marrero and Butson [1973] and from cyclic (0,-1,1)-matrix theory, Lee [1978] has shown how to construct complete sets of asymptotically orthogonal F-squares for some $n = 4t+2$ and for two symbols. The big

problem remaining is how to construct complete sets of orthogonal F-squares with more than the minimum number of symbols for any n. Mandeli and Federer [1977] have shown how to construct asymptotically complete sets of orthogonal F-squares for $n = 2s^m$, for s a prime power with s symbols. Federer [1975] and Anderson et al. [1974] have presented seven F(6;2,2,2) squares (i.e., three symbols each occurring two times in each row and in each column of the six by six square) and one latin square of order six which are mutually orthogonal. The treatment degrees of freedom in these eight squares account for 19 of the 25 row by column interaction degrees of freedom. Can one obtain additional F-squares with 6 degrees of freedom among symbols to complete the set? What is the totality of possible complete sets of orthogonal F-squares? What are the corresponding geometries for all possible complete sets of F-squares for all values of n and varying numbers of symbols.

6. Problem 6. Single degree of freedom contrasts and the analysis of variance procedure form important aspects of statistical analyses of data from an experiment. Concepts and uses related to these methods can become a powerful tool in mathematical research. If one has an n-row by n-column square, if one obtains the n-1 orthogonal contrasts among the rows, and if one obtains the n-1 orthogonal contrasts among the columns, then the $(n-1)^2$ single degree of freedom row by column interactions contrasts may be obtained from the Kronecker product of the row contrast matrix and the column contrast matrix. Hence, given the row and column contrasts the interaction contrasts are determined. Problem — given the $(n-1)^2$ single degree of freedom interaction contrasts, what row and column contrasts were used such that their Kronecker product produced the specified interaction contrasts? For n a prime power and for certain sets of interaction degrees of freedom, this problem is resolvable in terms of generalized interactions

(see Federer [1955] and Mandeli [1975]). In general, we do not know the answer. We should note also that the first row of both the row and column orthogonal contrast matrices is composed of plus ones.

The question arises as to why one would want this result. For the problem of obtaining complete sets of F-squares for $n = 6$ as described in the previous section (also see Federer [1975]), we have 19 of the $25 = (6-1)^2$ single degrees of freedom contrasts for the interaction. These are orthogonal to the row and column contrasts. Now, if one could determine which row and column contrasts gave rise to these 19, one could then use the remaining six degrees of freedom to construct F-squares to form a complete set of orthogonal F-squares of order six. It is possible that interactions obtained by Kronecker product may not be the correct approach. It may be that an addition system similar to that used for prime powers to obtain elements of a projective geometry should be used to obtain interaction contrasts (see Mandeli and Federer [1977]).

Another linear model result that would be useful in the construction of experiment designs is to know the conditions under which the following conjecture is true:

Conjecture: Let C_{ij}, $i = 1,2,\cdots,n$, $j = 1,2,\cdots,n-1$, be the coefficients of $n-1$ mutually orthogonal contrasts among n items. Then, for any j, say j^*, the set of k, $1 \le k \le n-1$, distinct contrasts in the set C_{ij^*}, $C_{ij}C_{ij^*}$, $j \neq j^*$, are mutually orthogonal. Also, for any two j, say j^* and j^+, the set of k, $2 \le k \le n-1$, distinct contrasts in the set C_{ij^*}, C_{ij^+}, $C_{ij^*}C_{ij^+}C_{ij}$, $j \neq j^+ \neq j^*$, are mutually orthogonal.

A start on the proof can be made by noting that $\Sigma_{i=1}^{n} C_{ij}C_{ij^*} = 0$, $j \neq j^*$, from the orthogonality of the C_{ij}, and hence is a contrast. From the Helmert contrasts, we observe that there are one to n-1 distinct contrasts formed, depending upon which j^* is selected. But, how does one prove that $\Sigma_{i=1}^{n} C_{ij^*}^2 C_{ij} = 0$ or that $\Sigma_{i=1}^{n}(C_{ij}C_{ij^*})(C_{ij}C_{ij^+}) = 0$ for $j \neq j^* \neq j^+$? One should note that the conjecture is not true for all orthogonal contrasts matrices; but when does it hold true?

The lack of linear model theory for this area is further exemplified by questions such as the following:

(i) How should one select the row (R_i) and column (C_j) contrasts and the treatment contrasts in order to obtain the simplest relation between $R_i \otimes C_j$ and the treatment contrasts from orthogonal F-squares?

(ii) What kind of contrasts should be formed by the interaction of treatment contrasts for two or more orthogonal F-squares?

(iii) If one interacts treatment contrasts from two orthogonal F-squares and if these contrasts are orthogonal to row, column, or both contrasts, what does this mean?

(iv) How does one use the concept of "closure under multiplication" as used by Mandeli [1975] for $n \neq$ a prime power?

The considerable amount of orthogonality among orthogonal contrasts in the analysis of variance should be exploitable to construct orthogonal F-squares. We need to know how to combine sets of orthogonal degrees of freedom to form latin squares and F-squares. Solutions of the above linear models problems should go far in helping with the construction problem of experiment designs.

7. Problem 7. In the optimum design problem, the dependent variable Y is a p-degree polynomial function of an independent variable X of the form $Y = f(X;\beta_1,\beta_2,\cdots,\beta_p)$ where the β_i are polynomial regression coefficients. For $a \leq X \leq b$, the factor space, how does one select values of the X variable to minimize the variances of the p estimated regression coefficients? For $p = 2$

$Y = \beta_0 + \beta_1 X$, the optimal design is to place one half, $N/2$, of the observations at a and the remaining $N/2$ at the point b. Results are available in the literature for various values of p and for k different values of X. Now, consider that we know that $Y = f(X;\beta_1,\beta_2,\cdots,\beta_p)$ goes through the point (X_0,Y_0). Two situations arise.

(i) $a \le X_0 \le b$

(ii) X_0 not in the factor space $a \le X \le b$.

Note also that the variance of Y at (X_0,Y_0) is zero and this immediately rules out the homoscedasticity property used by present researchers on the optimal design problem. Perhaps it would be reasonable to assume that homoscedasticity holds outside the interval $c \le X_0 \le d$, perhaps it can be assumed that the variance of Y is a monotone increasing function of X (e.g., $(X_i-X_0)^2\sigma^2$), or perhaps another situation for the variance of Y at some point X_i could be postulated. Then, for each of these cases, it would be necessary to develop optimal design theory. One result for $Y = \beta_0 + \beta_1 X$ given that this regression goes through the point (X_0,Y_0), e.g., the origin, is to place all of the N observations at the X_i in [a,b] the most distant from X_0. For three values of X, for quadratic regression, and for $a < X_0 < b$, is the optimal design to place $N(X_0-a)/(b-a)$ observations at a and $N(b-X_0)/(b-a)$ observations at the point b? What happens when X_0 is not in the interval [a,b]?

REFERENCES

[1] ANDERSON, D. A., FEDERER, W. T., and SEIDEN, E. (1974). On the construction of orthogonal F(2k,2) squares. BU-500-M in the Mimeo Series of the Biometrics Unit, Cornell University.

[2] ELFVING, G. (1952). Optimum allocation in linear regression theory. Ann. Math. Statist. 23, 255-262.

[3] ELFVING, G. (1956). Selection of nonrepeatable observations for estimation. Proc. Third Berkeley Symposium, Math. Statist. and Prob. 1, 69-75.

[4] FEDERER, W. T. (1955). Experimental Design — Theory and Application. Macmillan Company, New York. (Indian edition by Oxford and I.B.H. Publishing Company, Calcutta, India, 1967; copies may be purchased from author.)

[5] FEDERER, W. T. (1975). On the construction of F-squares and single degree of freedom contrasts. BU-566-M in the Mimeo Series of the Biometrics Unit, Cornell University.

[6] FEDERER, W. T. (1977). On the existence and construction of a complete set of orthogonal F(4t;2t,2t)-squares design. Ann. Statist. 5, 561-564.

[7] FEDERER, W. T., HEDAYAT, A., PARKER, E. T., RAKTOE, B. L. SEIDEN, E., and TURYN, R. J. (1970). Some techniques for constructing mutually orthogonal latin squares. Proc. Fifteenth Conference on the Design of Experiments in Army Research Development and Testing, ARO-D, Report 70-2, pp. 673-796. (Also MRC Technical Summary Report #1030, Mathematics Research Center, University of Wisconsin, 1971.)

[8] FEDERER, W. T., JOINER, J. R., and RAGHAVARAO, D. (1974). Construction of orthogonal series 1 balanced incomplete block designs — a new technique. Utilitas Mathematica 5, 161-166.

[9] FEDEROV, V. V. (1972). The Theory of Optimum Experiments. Academic Press, New York.

[10] HEDAYAT, A. (1971). A set of three mutually orthogonal latin squares of order 15. Technometrics 13, 696-698.

[11] HEDAYAT, A. and FEDERER, W. T. (1969). An application of group theory to the existence and nonexistence of orthogonal latin squares. Biometrika 56(3), 547-551.

[12] HEDAYAT, A. and FEDERER, W. T. (1970). On the equivalence of Mann's group automorphism method of constructing an O(n,n-1) set and Raktoe's collineation method of constructing a balanced set of ℓ-restrictional prime-powered lattice designs. Ann. Math. Statist. 41(5), 1530-1540.

[13] HEDAYAT, A., RAGHAVARAO, D., and SEIDEN, E. (1975). Further contributions to the theory of F-squares design. Ann. Statist. 3(3), 712-716.

[14] JOHNSON, D. M., DULMADGE, A. L., and MENDELSOHN, N. S. (1961). Orthomorphisms of groups and orthogonal Latin Squares I. Cana. J. Math. 13, 356-372.

[15] KIEFER, J. (1959). Optimum experimental designs (with discussion). J. Roy. Statist. Soc., B, 21, 272-319.

[16] KIEFER, J. (1961). Optimum designs in regression problems, II. Ann. Math. Statist. 32, 298-325.

[17] LEE, F-C, H. (1978). Sets of orthogonal F-squares for n = 4t+2 for 2t+1 symbols. Ph.D. Thesis, Cornell University.

[18] MANDELI, J. P. (1975). Complete sets of orthogonal F-squares. M.S. Thesis, Cornell University.

[19] MANDELI, J. P. (1978). On orthogonality of F-squares and F-hypercubes. Ph.D. Thesis, Cornell University.

[20] MANDELI, J. P. and FEDERER, W. T. (1977). On the existence and construction of orthogonal F-squares of order $n = 2s^p$, s a prime number. BU-622-M in the Mimeo Series of the Biometrics Unit, Cornell University.

[21] MARRERO, O. and BUTSON, A. T. (1973). Modular Hadamard matrices and related designs. J. Comb. Theory (A) 15, 257-269.

[22] RAKTOE, B. L. (1967). Application of cyclic collineations to the construction of balanced L-restrictional prime powered lattice designs. Ann. Math. Statist. 38, 1127-1141.

[23] WYNN, H. (1972). Results in the theory and construction of D-optimum experimental designs (with discussion). J. Roy. Statist. Soc., Ser. B, 34, 133-147.